A focused and coherent progression for fractions

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Focused and coherent

- Focused
  1. having the mind fixed on something
  2. not divided or scattered among several areas of interest or concern

- Coherent
  1. according to the rules of logic
  2. capable of being understood
  3. not having or showing any apparent conflict

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coherent

- Logically or aesthetically ordered or integrated
  - To integrate: to combine (two or more things) to form or create something
- Having clarity or intelligibility
- Having the quality of holding together or cohering
  - To cohere: to become united in principles, relationships or interest; to be logically or aesthetically consistent

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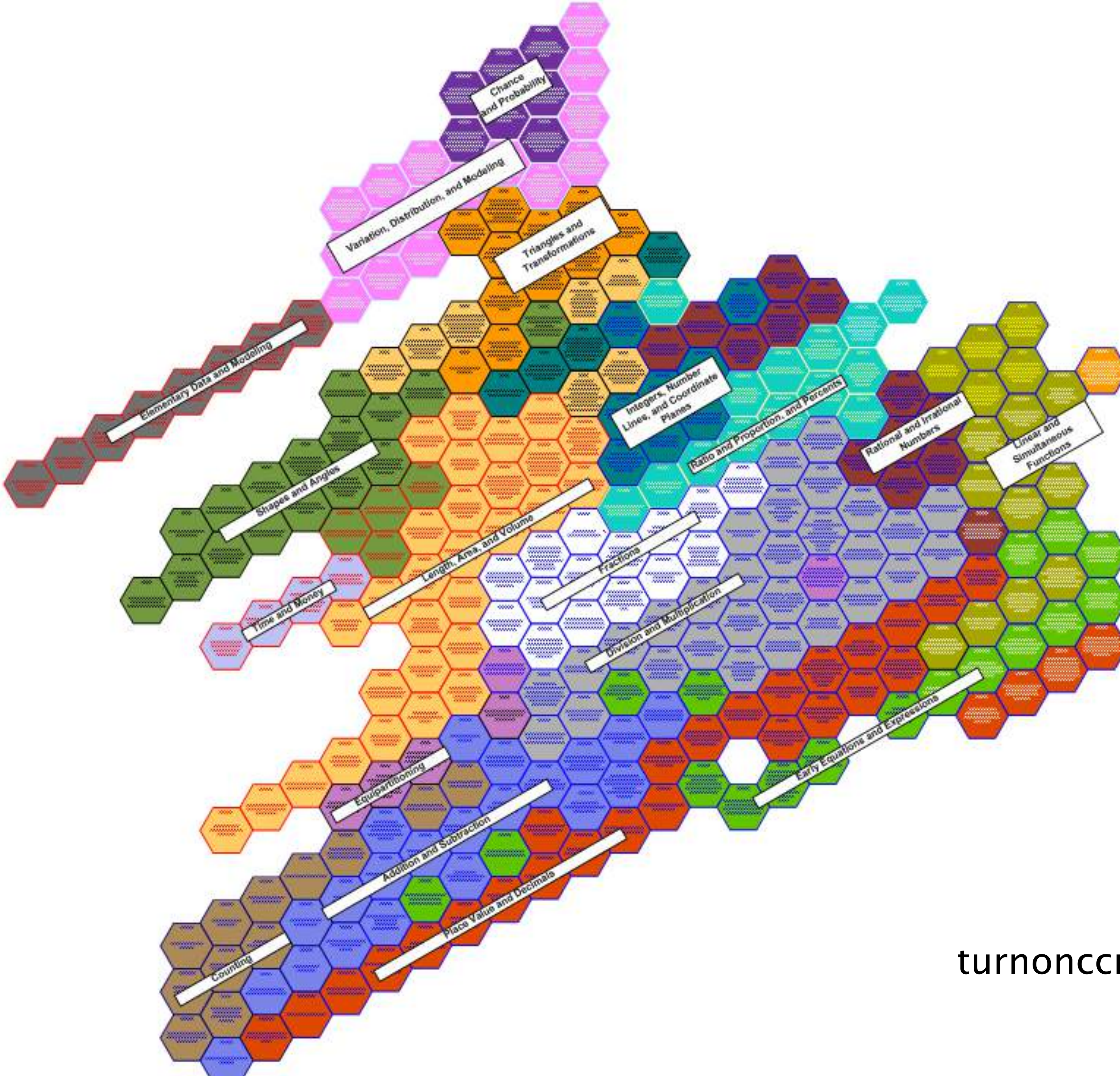
Full definition of coheret

Fractions in CCSS

- Laying foundations – equal partitioning
  - 1.G.3, 2.G.3 & 3.G.2
- Understanding fractions as numbers
  - 3.NF, 4.NF.A & 5.NF.3
- Understanding fraction arithmetic
  - 4.NF.B, 5.NF, 6.NS.A & 7.NS



turnonccmath.net

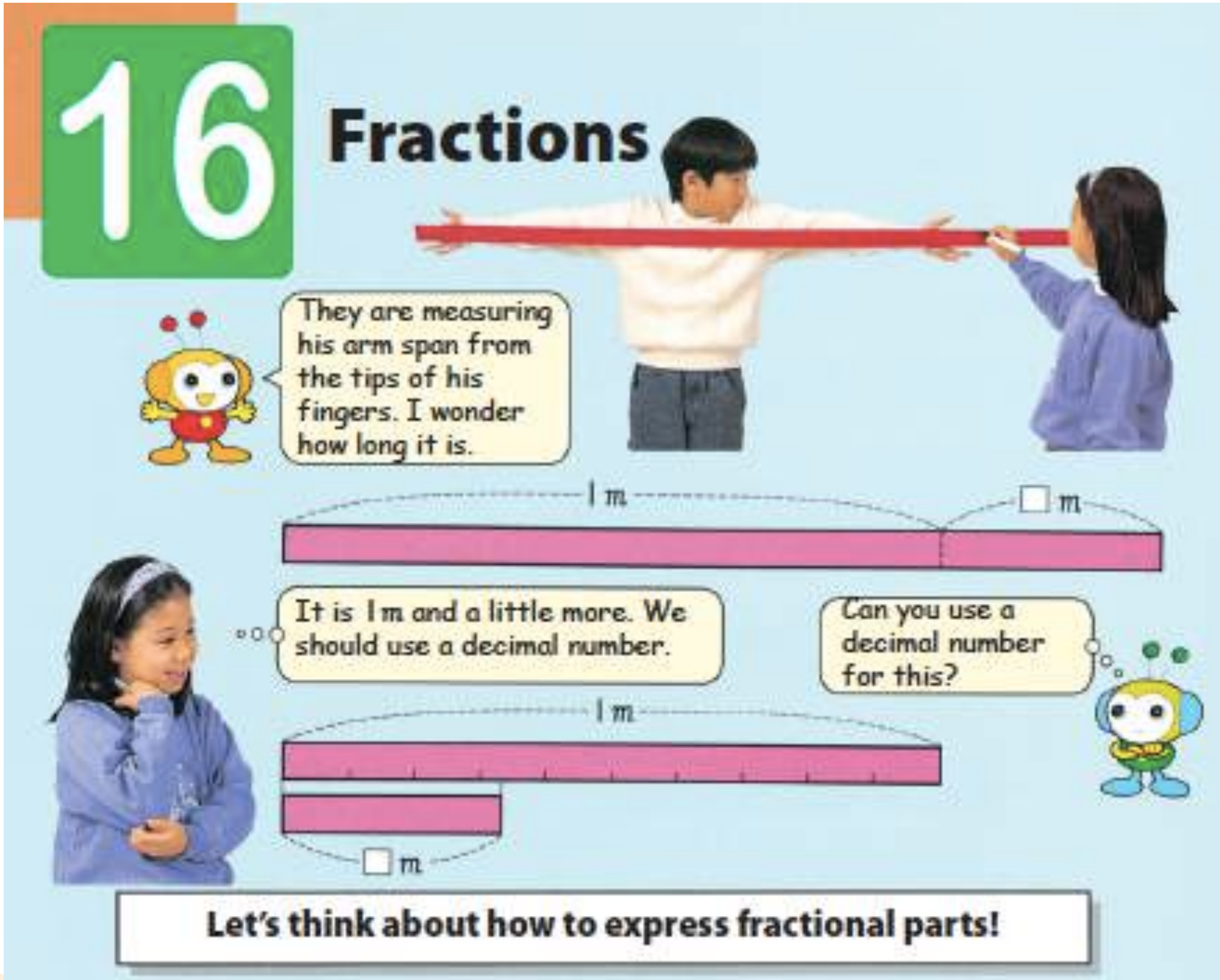
Focus on "understanding"

- Develop understanding of fractions as numbers (3)
- Extend understanding of fraction equivalence and ordering (4)
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers (4)
- Understand decimal notation for fractions, and compare decimal fractions (4)
- Use equivalent fractions as a strategy to add and subtract fractions (5)
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions (5)
- Apply and extend previous understandings of multiplication and division to divide fractions by fractions (6)
- Apply and extend previous understandings of operations with fractions to add, subtract, multiply and divide rational numbers (7)

Interpretations of Fractions
Japanese Course of Study: Elementary Mathematics

1. Expressing the quantity comprised of two of the three equal parts of a concrete object.
2. Expressing a measured quantity, such as ⅔ liters ⅔ meters.
3. Expressing twice the unit that is obtained when 1 is partitioned into three equal parts (i.e., the unit fraction ⅓).
4. Expressing the relative size of A when B is considered as 1, in usage such as "A is ⅔ of B."
5. Expressing the result (quotient) of whole number division, "2 ÷ 3."

Focusing on Measurement

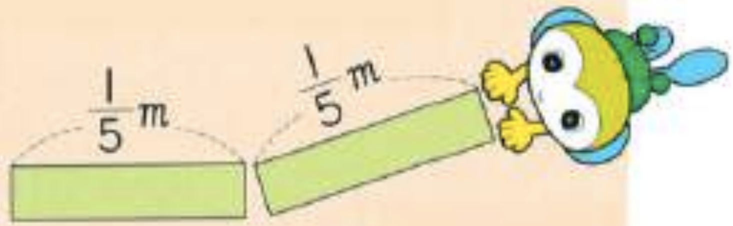


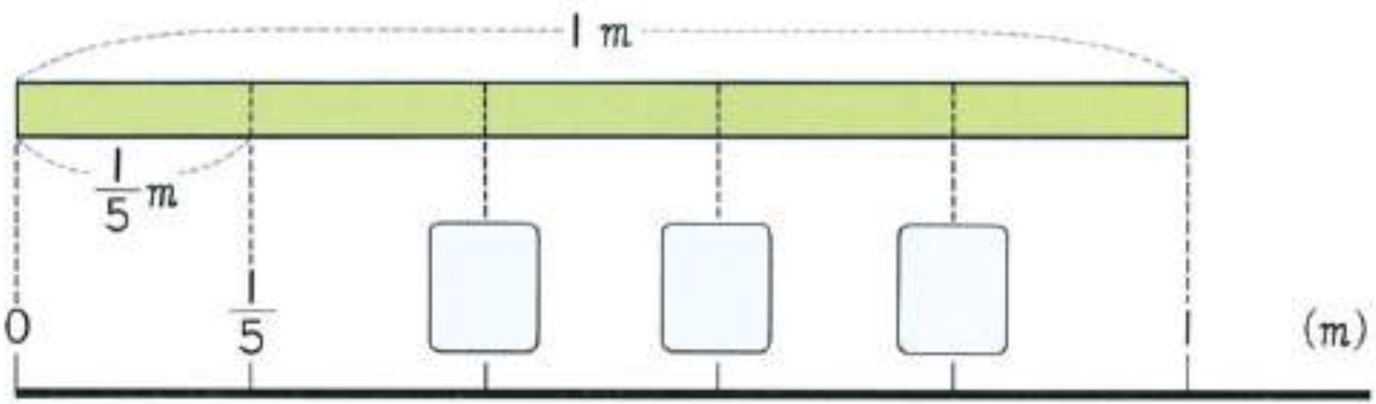
Tokyo Shoseki (2006), 3B, p. 57

Focusing on Measurement

1 ? How long are 2, 3, and 4 pieces of $\frac{1}{5}$ m?

Please express these as fractions on a number line.





Tokyo Shoseki (2006), 3B, p. 60

3.NF.A.1

Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

3.NF.A.3.d Comparing Fractions

Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.

Comparing Fractions

- Compare two fractions with the same denominator
  - Same denominator → Same-sized unit
  - The more units there are (i.e., the larger the numerator), the larger the fraction.
- Compare two fractions with the same numerator
  - Same numerator → Same number of units
  - The larger the unit (i.e., the smaller the denominator), the larger the fraction

Caution 1

- Compare two fractions with the same numerator
- Misconception: Given two proper fractions, the one with the smaller difference between the numerator and the denominator is larger.

Example

$\frac{2}{3} > \frac{2}{5}$ because $3-2 < 5-2$

Caution 2

- *"Recognize that comparisons are valid only when the two fractions refer to the same whole."*
  - Fractions as numbers → Whole = 1
  - Comparing fractions
    vs. Comparing fractional parts
  - "expressing the relative size of A when B is considered as 1, in usage such as 'A is ⅔ of B'"

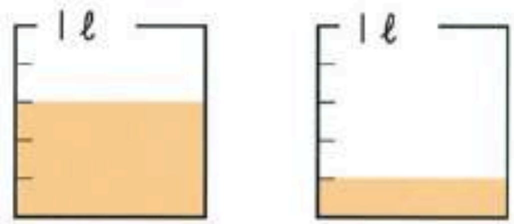
Addition/Subtraction

Addition and subtraction of fractions

3 There is $\frac{3}{5}\,\ell$ of juice in a carton and $\frac{1}{5}\,\ell$ in a bottle. How much juice is there altogether?

Let's think about how to calculate $\frac{3}{5} + \frac{1}{5}$.

1 How many $\frac{1}{5}\,\ell$ are in each $\frac{3}{5}\,\ell$ and in $\frac{1}{5}\,\ell$?



$\frac{3}{5}\,\ell \cdots \square\ \frac{1}{5}\,\ell$

$\frac{1}{5}\,\ell \cdots \square\ \frac{1}{5}\,\ell$

$$\frac{3}{5} + \frac{1}{5} = \square$$

Answer: $\square\ \ell$

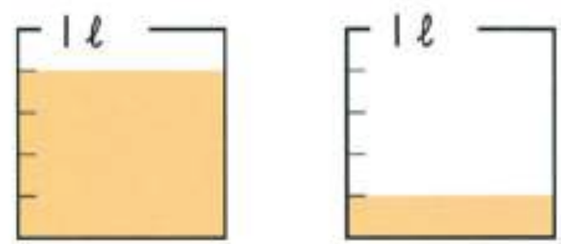
Tokyo Shoseki (2006), 3B, p. 62

Addition/Subtraction

5 There is $\frac{4}{5}\ \ell$ of juice.
If a girl drink $\frac{1}{5}\ \ell$ of juice, how much juice will be left?

Let's think about how to calculate $\frac{4}{5} - \frac{1}{5}$!

1 How many $\frac{1}{5}\ \ell$ are in each $\frac{4}{5}\ \ell$ and in $\frac{1}{5}\ \ell$?



$\frac{4}{5}\ \ell \cdots \square\ \frac{1}{5}\ \ell$

$\frac{1}{5}\ \ell \cdots \square\ \frac{1}{5}\ \ell$

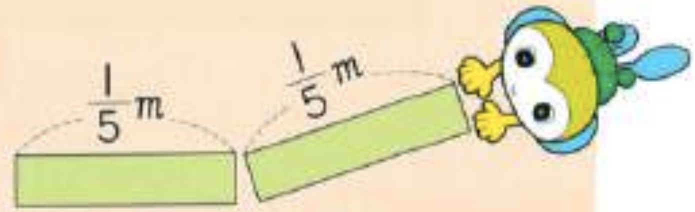
$\frac{4}{5} - \frac{1}{5} = \square$

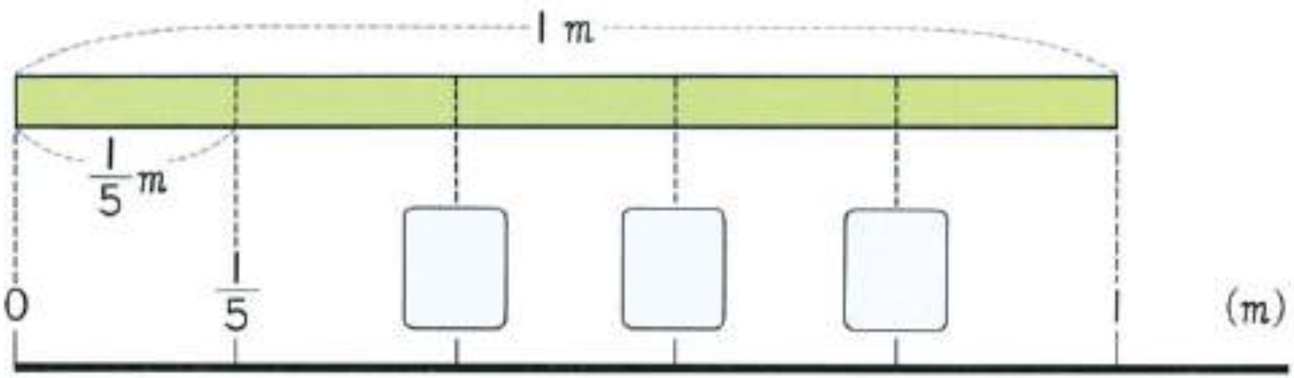
Answer: $\square\ \ell$

Tokyo Shoseki (2006), 3B, pp. 62 – 63

Focusing on number line

1 How long are 2, 3, and 4 pieces of $\frac{1}{5}$ m? Please express these as fractions on a number line.





Tokyo Shoseki (2006), 3B, p. 60

Focusing on number line

- 1 as the distance from 0 to 1, not the point.
- 1 as a whole → the interval (0, 1) is the whole

Multiplication and Division

- Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions (5)
- Apply and extend previous understandings of multiplication and division to divide fractions by fractions (6)
- Apply and extend previous understandings of operations with fractions to add, subtract, multiply and divide rational numbers (7)

Previous understanding of multiplication

- Equal sets – 3.OA
- Comparison – 4.OA
- Scaling (re-sizing) – 5.NF.5

Previous understanding of multiplication

- Equal sets – 3.OA
- Comparison – 4.OA

  Multiplying fractions by whole numbers

  $5 \times \frac{2}{3}$: 5 groups of $\frac{2}{3}$, or 5 groups of 2 $\frac{1}{3}$-unit
- Scaling (re-sizing) – 5.NF.5

  $\frac{2}{3} \times$ [ ]: $\frac{2}{3}$ groups of ???

  $\frac{2}{3} \times$ [ ]: $\frac{2}{3}$ of [ ]

5.NF.B.4.a

Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently as the result of a sequence of operations $a \times (q \div b)$.

Example

$\frac{2}{3} \times 5$: 2 parts of a partition of 5 into 3 equal parts, or $2 \times (5 \div 3)$

5.NF.3 → $5 \div 3 = 5/3$

4.NF.B.4 → $2 \times 5/3 = 10/3 = 3\frac{1}{3}$

Caution

What about $\frac{2}{3} \times \frac{4}{5}$?

2 parts of a partition of $\frac{4}{5}$ into 3 parts, or

$2 \times (\frac{4}{5} \div 3)$

$2 \times (\quad)$: 4.NF.4

BUT

$\frac{4}{5} \div 3$?

Division of Fractions

5NF.7:

Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions

$\frac{4}{5} \div 3 = (4 \times \frac{1}{5}) \div 3 = 4 \times (\frac{1}{5} \div 3)$

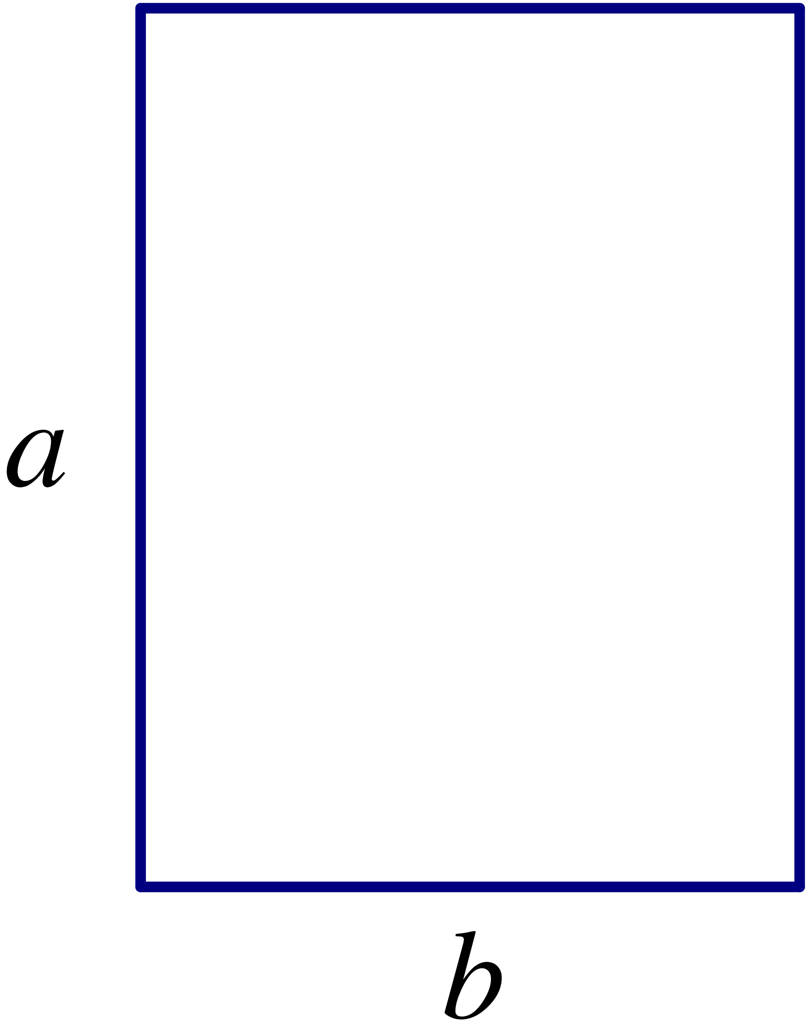
Challenge # 1: $(4 \times \frac{1}{5}) \div 3 = 4 \times (\frac{1}{5} \div 3)$

$(4 \times \frac{1}{5}) \div 3 = 4 \times (\frac{1}{5} \div 3)$?

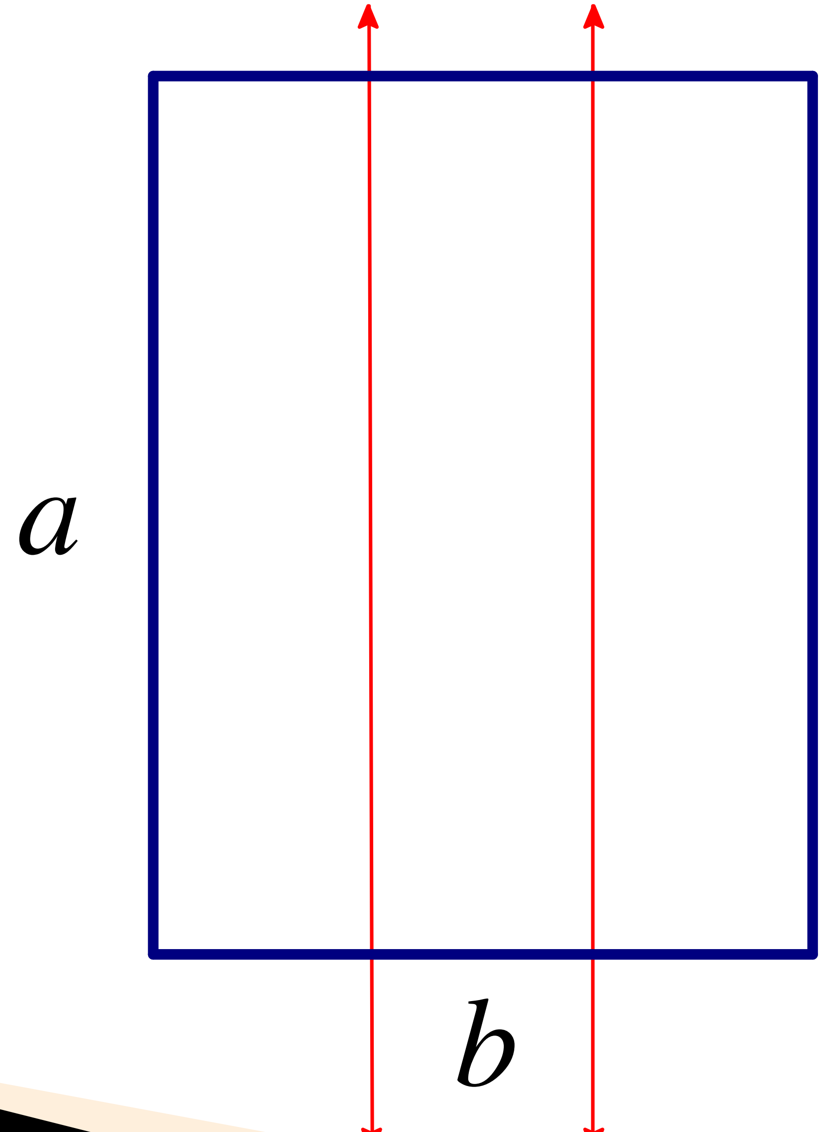
- $(20 \times 4) \div 2 = 80 \div 2 = 40$
- $20 \times (4 \div 2) = 20 \times 2 = 40$

- But
- $(20 \div 4) \times 2 = 5 \times 2 = 10$
- $20 \div (4 \times 2) = 20 \div 8 = 2.5$

$$(a \times b) \div c = a \times (b \div c)\ ?$$



$$(a \times b) \div c = a \times (b \div c)\ ?$$



Division of Fractions

5NF.7:

Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions

$\frac{4}{5} \div 3 = (4 \times \frac{1}{5}) \div 3 = 4 \times (\frac{1}{5} \div 3)$

Challenge 2: How can students reason $\frac{1}{5} \div 3$?

Division of Fractions: Alternate Route

Dividing fractions by whole numbers

- $\frac{4}{5} \div 2 = ?$

  If 4 1/5–units are partitioned into 2 groups, how much is in each group?

  $4 \div 2 = 2$ 1/5–units in each group.

  $\frac{4}{5} \div 2 = (4 \div 2)/5$

Division of Fractions: Alternate Route

- $\frac{3}{5} \div 2 = ?$

  If 3 1/5-units are partitioned into 2 groups, how much is in each group?

  $3 \div 2 =$ not a whole number!

- But, $\frac{3}{5} = 6/10$

  If 6 1/10-units are partitioned into 2 groups, how much is in each group?

  $6 \div 2 = 3$ 1/10-units in each group

  $\frac{3}{5} \div 2 = 6/10 \div 2 = (6 \div 2)/10 = 3/10$

Division of Fractions: Alternate Route

In general,

$$a/b \div n = a/(b \times n)$$

Dividing by Fractions

- Meaning of division
  - Partitive division

    Product ÷ (number of groups) = (group size)

  - Quotitive division

    Product ÷ (group size) = (number of groups)

Measurement division

- 9/5 ÷ 3/5: How many 3/5 are there in 9/5?

  9 ÷ 3 = 3

  9/5 ÷ 3/5 = 3

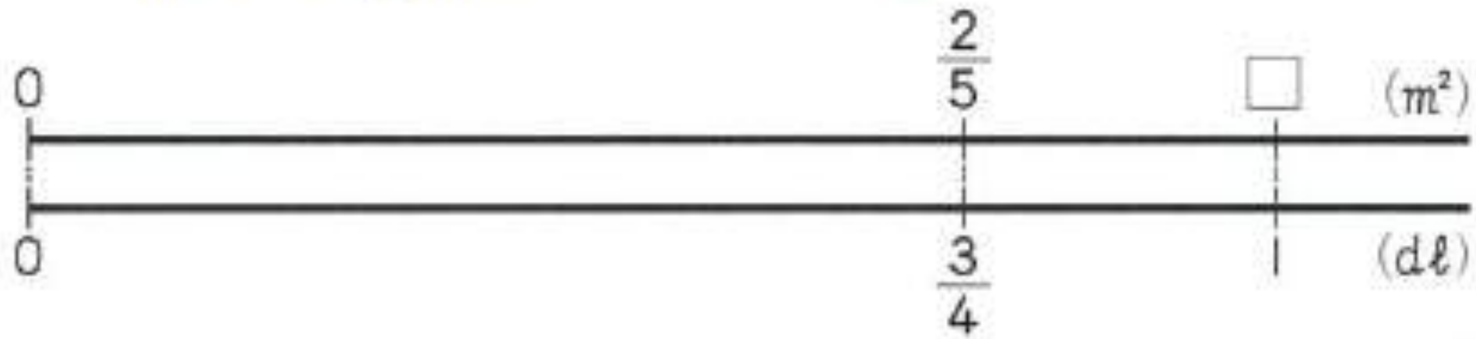
- 8/5 ÷ 3/5: How many 3/5 are there In 8/5?

  8 ÷ 3 = 2 rem. 2

  What do we do with the remainder?

Dividing by Fractions

1 With $\frac{3}{4}$ dℓ of paint you can paint $\frac{2}{5}$ m^2 of board. How many m^2 can you paint with 1dℓ of paint?

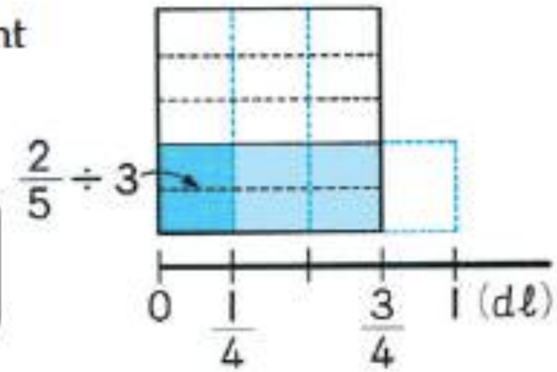


$\frac{2}{5} \div \frac{3}{4} = ?$

Dividing by Fractions

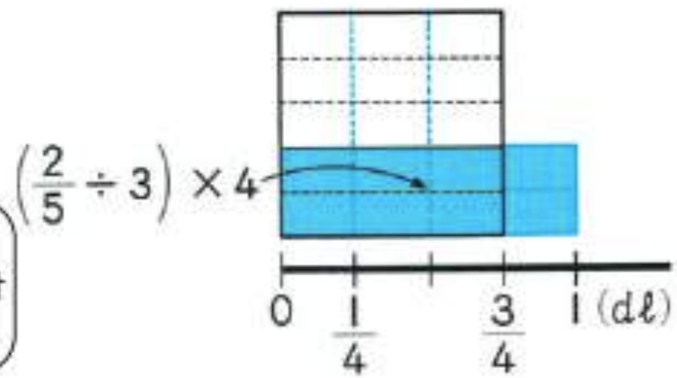
You can calculate the area that can be painted with $1 d\ell$ by first finding the area that can be painted with $\frac{1}{4} d\ell$ and then multiplying it by 4.

2 How many m^2 can you paint with $\frac{1}{4} d\ell$?



Divide $\frac{2}{5} m^2$ into 3 equal parts and then . . .

3 How many m^2 can you paint with $1 d\ell$?



Multiply the area that you can paint with $\frac{1}{4} d\ell$ of paint by 4, then . . .

Dividing by Fractions

$\frac{2}{5} \div \frac{3}{4}$

4 parts of a partition of $\frac{2}{5}$ into 3 parts,

or

$4 \times (\frac{2}{5} \div 3)$,

Or

$\frac{2}{5} \times 4/3$

$a/b \div c/d = a/b \times d/c$

- Division is the same as multiplication by the reciprocal of the divisor.
- "Same" operations should involve "same" reasoning → ratio/rate reasoning.
- Multiplication and division are special cases of proportion problems.
  - Multiplication $A : 1 = ? : B$ $B \times A = ?$
  - Measurement $A : 1 = B : ?$ $B \div A = ?$
  - Partitive $? : 1 = A : B$ $A \div B = ?$

A focused and coherent progression

builds and is built on students' disposition to look for and make use of

- structures (MSP 7) and
- repeated reasoning (MSP 8).