The Relationship Between Remainder and Divisor
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Team Members

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Lesson Date:	Instructor:	Grade Level:
5/15/19	Joshua Lerner	3

1. Title of Lesson

The Relationship Between Remainder and Divisor

2. Research Theme

The long-term goals of our students

We want to help our students approach mathematics as a process of inquiry, rather than a process of answer-getting. Evidence of this development would include: students justifying and critiquing mathematical arguments; asking each other questions to understand different problem-solving strategies; acknowledging errors as learning opportunities; and contributing to a classroom culture that is positive, collaborative, and encouraging for all.

We hope to achieve this goal by improving our practice of teaching mathematics through problem solving, with a focus on student discussion. Student discussion will be fostered by strategic and consistent notebook use, frequent opportunities to justify and critique mathematical strategies both in small and large groups. This will be intentionally taught to ultimately generalize the value of the process of problem solving.

3. Background and Research on the Content

- Why we chose to focus on this topic - for example, what is difficult for our students, what we noticed about student learning
- What resources we studied, and what we learned about the content and about student thinking
- What we learned from studying our own curriculum and other resources

Choosing our topic

For the past few years, teachers from the 3rd grade team at Peirce have considered teaching Unit 7 (Tokyo Shoseki, Grade 3) on division with remainders (DWR). We have seen that students develop a good understanding of division over the course of the year, especially as they relate their knowledge of multiplication to learning their division facts. It has always been our belief that, if exposed to a clear progression of learning, our students would pick up the topic of DWR quite readily. However, due to the amount of important topics to be introduced and taught in 3rd grade, we have never made time to teach this unit.

This research lesson provides an opportunity for our team to study Unit 7 and carefully teach this unit to one 3rd grade class in the spring. Our goal is to show that 3rd grade students are in fact able to understand important points related to DWR, even though the Common Core State Standards (CCSSM) implies DWR should be introduced in 4th grade. We expect our students to learn how to perform division calculations with remainders, how to interpret the relationship between the remainder and divisor, and how to handle the various meanings of remainders in context. As we have studied this topic, we have also developed the belief that DWR merits its own thorough learning progression, which we explain below.

When should we teach DWR?

The Common Core does not mention remainders in the Grade 3 division standards. In Grade 4, the introduction of DWR is merely implied rather than included as its own unique standard. The first time we read about remainders is in a standard about multi-step word problems using the four operations, including problems in which "remainders must be interpreted" (4.OA.A.3). Meanwhile, 4.NBT.B.6 states that students should "find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors..." The collapsing of DWR and multi-digit whole number division into a single standard creates pedagogical concerns that are hard to reconcile. Should DWR be taught before multi-digit division with simple numbers? Or should it be taught within the lens of multi-digit division, such as when some tens are left over and regrouped with the ones to be divided again?

We studied many Grade 3 and Grade 4 math curricula and found these questions to be unresolved. For example, Go Math and EngageNY both include DWR lessons at the start of broader units on multi-digit division. However, in each curriculum, only two lessons focus on what a remainder is and how to calculate division with a dividend that is not a multiple of the divisor. Subsequent lessons focus on remainders of tens, hundreds, and thousands and how these can be combined with the next smallest unit and further subdivided. Meanwhile, in Everyday Math, students first see remainders in

Grade 3 in a single lesson with relatively simple numbers (e.g., $46 \div 10$). However, the next lesson devoted to DWR does not appear until Grade 4, when students are asked to solve problems such as $124 \div 8$ and $348 \div 16$ as well as consider what to do with the remainder within the context of the problem. Our team thought the progressions of learning for DWR within these curricula left much to be desired. Was it possible that the haziness around this instruction stemmed from the ambiguity within the CCSS itself? Believing that DWR merited its own clear sequence of instruction, we went back and studied another important facet of this topic: the contexts of partitive and quotitive division.

Partitive and quotitive division

To review, partitive division (or fair-share division) involves finding the size of a group, given the total amount and the number of groups. For example: *If there are 12 cookies to be shared among 3 students, how many cookies will each student get?* Quotitive division (or measurement division) involves finding the number of groups, given the total amount and the size of each group. For example: *There are 20 strawberries. Each student will get 5 strawberries. How many students can get strawberries?*

Why is it important for us to know both of these division contexts? For one, the two types of division situations can feel very different to students. It's an important mathematical idea that the same operation can be used for both types of situations. Second, knowledge among teachers of both partitive and quotitive division is essential in ensuring that students are exposed to a wide range of division contexts and problems. This helps students develop a comprehensive picture of what division means from the outset. Research has shown that an overreliance on partitive division models in the early grades can severely limit older students' abilities to correctly solve division problems involving fractions and decimals. Finally, armed with this knowledge, teachers can make strategic choices about whether to use partitive or quotitive division contexts to introduce new ideas related to division, such as DWR.

Introducing DWR - partitive or quotitive?

Although partitive division and the concept of fair sharing is often used to introduce division in Grade 3, our argument is that quotitive division is the better context in which to introduce DWR. An important reason for this is because quotitive division problems force students to approach a leftover amount as a fixed quantity to be considered and discussed. Consider the problem from the first lesson in our unit: *There are 14 puddings. When 3 are given to each person, how many people can get pudding?* The remainder is 2 puddings. On the other hand, imagine if the context were partitive, with 14 puddings shared among 3 people. Students may wish to further share the 2 puddings by subdividing them evenly among the 3 people. The meaning of the remainder becomes muddled when it can be subdivided and shared until it no longer exists as a leftover amount.

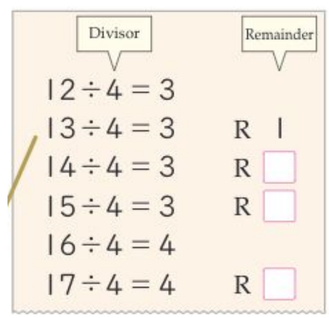
Some argue that teachers can stick with partitive division and still get around this confusion by simply choosing items that students cannot further subdivide (like dominoes, or cats for that matter). However, we argue that, even with this choice, partitive division is not the way to introduce DWR. When given a partitive context for $14 \div 3$, a common approach for students is to draw 3 circles and then draw one object at a time in the circles until the starting amount is used up. It's true that students are making 3 groups of 1, then 3 groups of 2, etc., as they perform this procedure. But are students thinking in terms of equal groups? They may simply draw dots and count 1, 2, 3, ... until they reach 14. Then they go back and check the number of dots in each circle. The equal groups they created along the way (the multiplication facts of 3 x 1, 3 x 2, etc.) are no longer visible as evidence of their process.

In contrast, quotitive division leads students to think in terms of equal groups in a natural way. To represent the problem, students may draw a diagram to show a certain number of equal groups or they may skip count by 3. Both approaches point to the relationship of division to multiplication. As they model the problem, students are quite concretely modeling 3×1, 3×2, 3×3, and 3×4 before seeing that there is not enough of the starting amount to make 3×5. This leaves 2 leftover. This connection to the basic multiplication facts becomes instrumental as students develop fluency in calculating DWR.

We were surprised to find that no other curriculum we studied introduced DWR using a quotitive context. In Go Math (Grade 4), students share 28 dominoes among 3 people. Students are instructed to draw 3 circles and to draw dots into each circle until all dominoes are used up. Practice problems for this unit include partitive and quotitive examples, but the difference between the two contexts is not explored. In EngageNY (Grade 4), students are asked to find the size of the teams when there are 13 students divided into 4 equal teams. Later within the same phase of the lesson, students are also asked to consider the situation if there were 13 students and exactly 3 students are needed on each team. In Everyday Math, partitive and quotitive contexts are both used within the lesson that introduces DWR in Grade 3.

The relationship between divisor and remainder

The goal of today's research lesson is for students to understand the relationship between divisor and remainder in any DWR problem. For any division problem with a remainder, the remainder increases by 1 as the dividend increases by 1 but never reaches the size of the divisor. At this point, a new group would be formed thus eliminating the remainder. As adults, we seem to know this intuitively. Yet students can benefit greatly from a systematic exploration of this mathematical rule.



Why? For one, understanding this facet of DWR will help students get DWR questions right. A remainder that is greater than the size of the divisor can be distributed again, creating a new group and a new remainder – the difference between an incorrect and correct answer on the test. Second, this reasoning (loosely, "How close can I get before I surpass the dividend? And how much will be left?") is the basis of most strategies used when approaching multi-digit division. Seeing subsequent calculations laid out in this way can also help students see why it is possible to check their work for any division problem by calculating *quotient* $\times$ *divisor* + *remainder* = *dividend*. And more broadly, the pattern created when exploring this feature of DWR is a beautiful representation of "repeated reasoning" as described in Standard for Mathematical Practice #8.

Despite the importance of this topic, we did not find coherent instruction of the relationship between divisor and remainder in any other curriculum we studied as part of our research process. We were drawn to this lesson because of our own interest in the mathematics involved and the opportunity it gave us to carefully design a lesson to show students this topic for the first time.

A clear progression of learning for DWR
We believe that by the end of 3rd grade, students are ready to take on DWR, given a few key requirements. First, DWR should be introduced using a quotitive division context with simple numbers. As the unit progresses, students should be exposed to many quotative and partitive division problems and have the opportunity to explore important reasoning around DWR, such as the relationship between the remainder and divisor as well as the meaning of the remainder in context. Finally, students should study DWR through a sufficient number of lessons to build procedural fluency from their conceptual understanding. We look forward to seeing how our students respond to the instruction we have designed.

4. Rationale for the Design of Instruction

- IF we teach this topic by doing X, THEN students will better understand Y.
- IF we structure the lesson as X, THEN we will observe Y. *(Research Theme)*

If we give students their own division problems with divisors of 4 and then ask them to organize them and discuss what they notice, then students will engage in active sense-making of the relationship between remainder and divisor. This will lead students to better understand the idea that if R ≥ divisor, then another group can be made.

If we use discussion techniques such as think time, revoicing, asking students to restate their classmates' reasoning, and prompting students to ask for comments and questions, then we will observe increased engagement and more equitable participation.

5. Goals of the Unit

- Students expand their understanding of division to include cases of division with remainders (DWR).
- Students can calculate DWR using a related multiplication fact.
- Students understand the size relationships of remainders and divisors.
- Students consider the meaning of the remainder in context and use this understanding to find the answer to the problem.

6. Unit Plan

The lesson sequence of the unit, with the task and learning goal of each lesson.

Lesson	Learning goal(s) and tasks
1	**Lesson Goal:** Extend the understanding and use of division to include cases where equal groups cannot be made -- the dividend is not a multiple of the divisor. Understand how to perform calculations in cases of division that result in a remainder (quotitive division). **Task:** There are 14 puddings. When 3 are given to each person, how many people can get puddings?
2 Research Lesson	**Lesson Goal:** Understand that the remainder, if not zero, should always be less than the divisor. **Task:** There are 13 apples. If 4 apples are put in each carton, how many cartons can you make? How many apples will be left?
3	**Lesson Goal:** Perform division calculations resulting in a remainder (partitive division). **Task:** There are 16 flower seeds. If we divide them evenly among 3 people, how many seeds will each person get? How many flower seeds will be left?
4	**Lesson Goal:** Understand how to check the answers of division calculations resulting in a remainder. **Task:** There are 23 sheets of colored paper. If we give 6 sheets to each person, how many people can we give the paper to? How many sheets will be left? **Practice:** Students practice solving and checking division problems.

5	**Lesson Goal:** Understand the remainder in problem situations (cases in which the quotient + 1 = answer). **Task:** There are 23 children. They are divided into groups to ride boats that can hold 4 passengers each. How many boats do we need if everyone rides in the boats?
6	**Lesson Goal:** Understand the remainder in problem situations (cases in which the quotient = answer; remainder is disregarded). **Task:** We have 30 flowers. We are making bouquets that have 4 flowers each. How many bouquets with 4 flowers can we make?
7	**Lesson Goal:** Develop procedural fluency with division problems involving remainders. **Practice:** Students practice various division problems with remainders.
8	**Lesson Goal:** Deepen understanding of math content. **Practice:** Students answer "Mastery Problems" involving DWR.

7. Relationship of the Unit to the Standards

- How the learning in the unit relates to the grade-level standards.
- How the learning in the unit relates to prior standards and future standards.

Prior learning standards that unit builds on	Learning standards for this unit	Later standards for which this unit is a foundation

CCSS.MATH.CONTENT.3.OA.A.2 Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.* CCSS.MATH.CONTENT.3.OA.A.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.1 CCSS.MATH.CONTENT.3.OA.A.4 Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = _ ÷ 3, 6 × 6 = ?*	CCSS.MATH.CONTENT.4.OA.A.3 Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. CCSS.MATH.CONTENT.4.NBT.B.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.	CCSS.MATH.CONTENT.5.NBT.B.6 Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

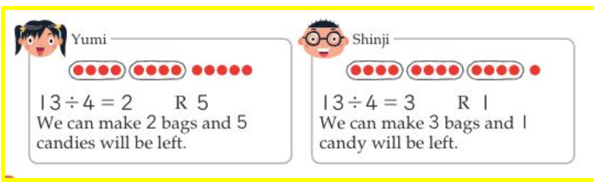
8. Goals of the Research Lesson

Content goal: Understand that the remainder, if not zero, should always be less than the divisor.

Language targets:
*Language supports for EL's described in Column 2.
- Explain, connect, and ask questions about mathematical ideas (speaking)
- Identify relationships between mathematical ideas as explained orally (listening)

9. Research Lesson Plan

Learning task and activities, anticipated student responses, key questions or comparisons that will build insights	Teacher Support	Assessment (Points to Notice)
Introduction Teacher invites students to look back in their notebook and share what was learned in yesterday's lesson with a partner. Students read selected reflections to the class. Explain to the class that today we are going to see what else we can find out about division with remainders.	Have several reflections from previous lesson visible so that students can follow along visually.	Do students understand that all numbers can be divided, even if the dividend is not a multiple of the divisor? Do students use precise vocabulary?
Posing the Task Concretely model grouping 4 apples together -- the size of the group in today's problem. Then pose the problem. **There are 13 apples. If 4 apples are put in each carton, how many cartons can you make? How many apples will be left?** As a class, write the division number sentence to represent the situation: **13 ÷ 4 =**	Make students aware that there will be two parts to today's lesson. Have problem statements pre-printed for select students to glue into notebooks.	Is the supporting student understanding of the context? Is there an entry point to the problem for all students?

Individual Problem Solving Anticipated Student Responses #1 (“Shinji”). 13 ÷ 4 = 3 R 1 - diagram of circled groups, and/or - skip counting - multiplication facts $1 \times 4,\ 2 \times 4,\ 3 \times 4,\ \ldots$ #2 (“Yumi”). 13 ÷ 4 = 2 R 5 #3. 13 ÷ 4 = 3 [2 groups of 4, 1 group of 5]  		What are the primary strategies that students are using to solve this problem? (skip counting, facts, diagrams, ect.)
Comparing and Discussing, including Teacher Key Questions Compare “Shinji” and “Yumi” side by side. Model Yumi’s idea first. Elicit feedback from students. Possibly have students model placing apples into cartons. Key questions: • What is different about these solutions and strategies? • What do you notice about the size of the remainders? • For which response can we still make another group of 4 apples? • What would you recommend that [Yumi] do next to finish the problem? Use “Yumi” to clarify that the remainder should be smaller than the divisor. Model this.	If many students show #3, remind the class that the groups must be equal size. If no one shows “Yumi,” show an example of this response to the class and ask what they notice.	Do students recognize how the two ideas are different? Do students show understanding that as many groups as possible should be made? How do students articulate the connection between the size of the remainder and divisor?

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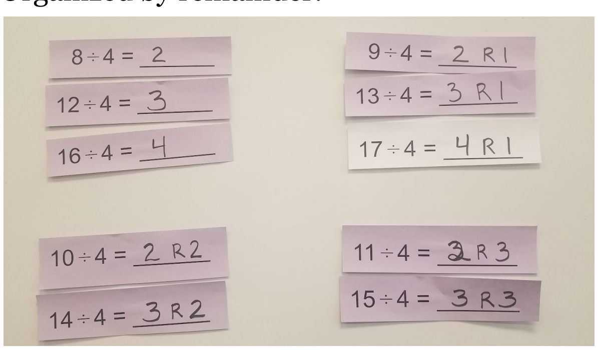
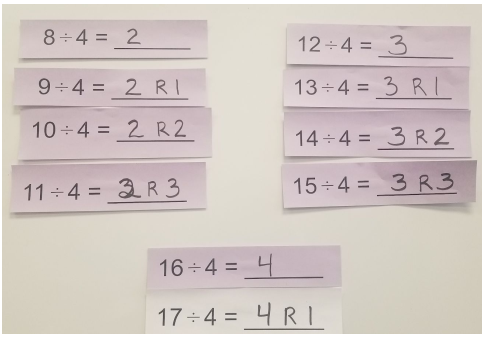
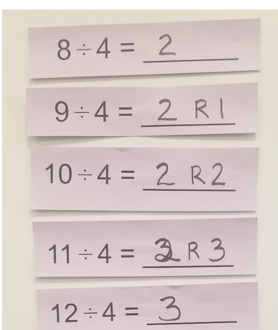
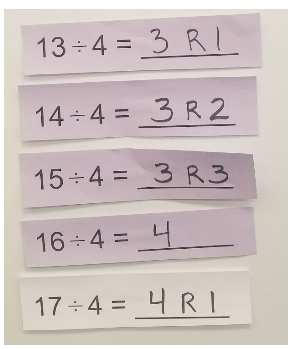
Posing the Task We notice that Shinji's remainder was correct because we couldn't make another group. The remainder (1) was smaller than the group size (4). However, Yumi's remainder was too large because we could make another group. The remainder (5) was larger than the group size (4). It seems like the remainder size has to be smaller than the group size (divisor). Will this always be the case? **Let's explore how the size of the remainder and the size of the divisor are related.** Explain that each partnership will get a new set of numbers to divide. Use the example of 13 ÷ 4 = 3 R 1 to model how it would be written on the strip.	Refer to the divisor as "size of the group" to help students understand the problem in this context. Label the numbers in 13 ÷ 4 = 3 R 1 on the board as an anchor for correct use of vocabulary.	
Individual Problem Solving Anticipated Student Responses Students talk with their partners and solve the division problem on their strip. The strips will have problems ranging from $8 \div 4$ to $22 \div 4$. Students can work in their journals or use counters at their table to calculate. Students can place the strip on the board as they finish.	If partners have a wrong answer on their strip, refrain from intervening. This can be addressed through discussion with classmates. If students don't know how to get started, refer them back to strategies shown on the board for the earlier problem. Strips for students modeling 18 divided by 4 to 22 divided by 4 will	What strategies are being used by students to solve this problem? Are students making as many groups as possible?

	have stickers on them. These students will be told to hold on to their strips, to be used later in the discussion.	
Comparing and Discussing, including Teacher Key Questions Once all strips are on the board: - What do we notice about the size of the remainder and the size of the divisor? - Let's check to see if all of the remainders are less than the divisor. - Were we right that the remainder has to be smaller than the divisor? - How do we know this will always be the case? To organize the strips on the board: ● How could we organize the strips to see relationships between the numbers? ● Why did you choose to organize them this way? Organized by remainder:  ● What do the facts with no remainder have in common?	First have students turn and talk to their neighbors about possible ideas for organization of strips. For efficiency, the student can explain how to move the strips while the teacher moves them. Students will use a pointer to point to the strips as they explain what they notice. If students do not use division vocabulary correctly (i.e., they say "first number" instead of dividend), refer them back to the labels/anchor from the first part of the lesson.	How did students choose to place the strips on the board? Are students excited to organize the strips? Which method of organization created a clearer understanding for students to visualize the relationship between divisor and remainder? Do students show understanding that the remainder is always smaller than the divisor,

• How come we see groups for no remainder, R1, R2, and R3--but not R4? Organized by number of groups (quotient):  (Questions under the next example apply to this one as well. See the next section below.) Organized by increasing dividend (cut to save space here):   • What do you notice about the number sentences? Anticipated Student Responses: • It looks like the remainder always increases by 1. There is a pattern to the remainder as the dividend increases by 1: 1, 2, 3, none • The ones with no remainder are the facts that we know. • I notice the remainder never reaches 4. • I notice that the remainder is never	Go back to the apple context (i.e., if R were greater than 4, then another group of apples could be made).	based on the repeated pattern on this list?

bigger than the divisor. I think that's because we would make a new group. Clarify that this repeating pattern helps us see that the remainder will always be less than the divisor. Invite students with strips $18 \div 4$ through $22 \div 4$ to confirm that the pattern continues.	To make thinking visible and lasting, keep a list of observations on the board throughout the discussion.	
Summing Up So it seems that we were right: no matter how high the dividend gets, the remainder will always be less than the divisor because, otherwise, another group could be made. We have a few more classmates that continued the pattern, let's see if it continues to work as the numbers get bigger. <Summary> If the remainder is larger than the divisor, then another group can be made. Therefore, the remainder should always be smaller than the divisor. <Reflection> Today I learned, ____ [Choose one of today's observations and explain it...]	Students with strips modeling 18 divided by 4 to 22 divided by 4 have stickers bring up strips.	

10. Points of Evaluation (Assessment)

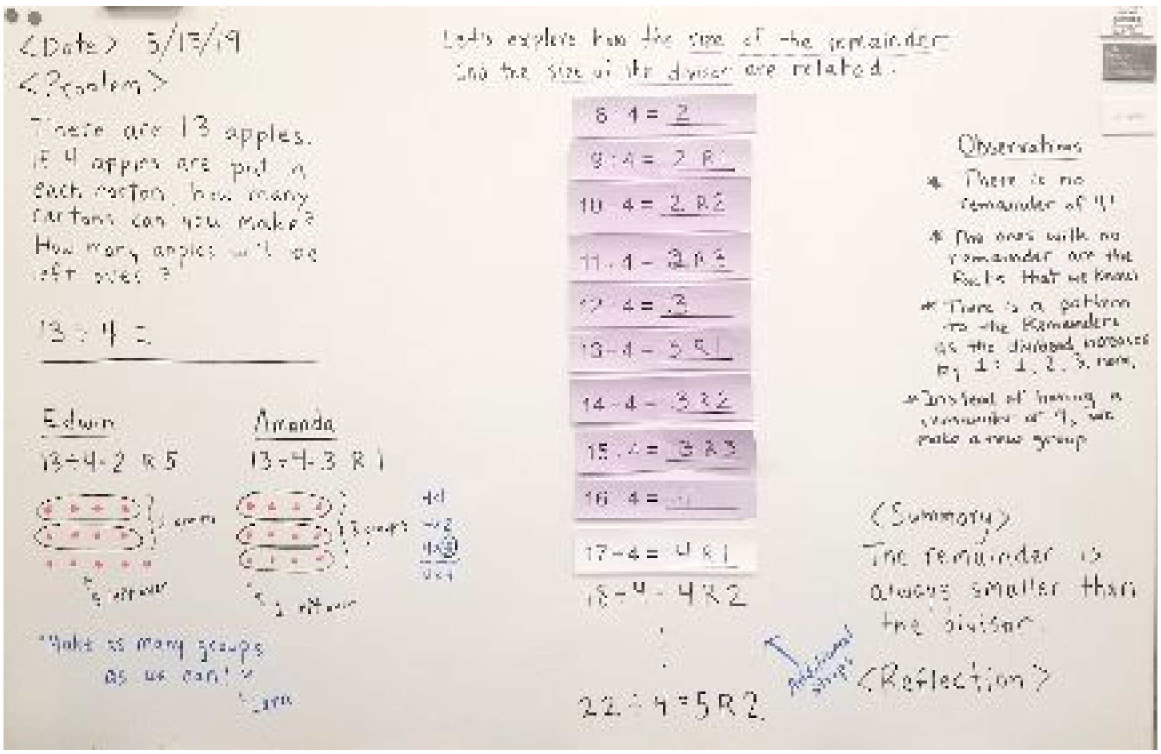
Prompts to focus observation and data collection.

Do students understand that R must be less than the divisor because, if R is equal to or

greater than the divisor, then another group can be made?

Do class discussion techniques help students engage deeply with the content and participate equitably?

11. Board Plan



12. End of Cycle Reflection

What Did We Learn? (to be filled out after the post-lesson discussion)

<u>Partitive vs quotitive:</u> A major point from the final comments had to do with Valeny's mistake at the start of the lesson. The class had a productive discussion about how she could fix her error (from 13 ÷ 4 = 2 R 5 to 13 ÷ 4 = 3 R 1). She eventually fixed it herself, which we thought was powerful. Dr. Takahashi's comment, however, was that the root of her error was never addressed. She solved by drawing 3 circles and then fairly sharing

apples into each circle. In other words, she misunderstood the context and did partitive division instead of quotitive division. Without addressing this error in the lesson, Valeny (and probably others) would continue to misunderstand division contexts throughout the unit, as these students probably only understand division as partitive.

The next day, Mr Lerner taught a "re-engagement lesson" with the class, in which Valeny's error was once again discussed, alongside Becky's correct method and response. This time, the class discussed the root of the error. Mr. Lerner asked the students to explain not only why the final answer was incorrect, but why the solving strategy did not match the problem. This was a productive discussion: many students were able to articulate that Valeny was trying to find how many go in a group, even though the problem asked for how many groups. Ms. Zisook then led the class through some "bare number" practice problems. However, she encouraged the students to approach these problems by thinking of them as quotitive situations (e.g., 26 ÷ 5 … think: "I'm making groups of 5. How many groups can I make. How many will be left?). We thought this day was a good example of building procedural fluency from experiences that are grounded in conceptual understanding.

<u>Remainder must be lower than divisor</u>: Another question we found ourselves wondering about is whether students truly understood that the remainder must always be lower than the divisor because otherwise another group could be formed. For example, during the discussion, many students were saying that the remainder "never gets to 4" or the remainder "always goes back to 0 after R 3" -- but they had a hard time articulating that this was because the quotient (number of groups) goes up 1 at this point. We do think that using the strips of division calculations in the way we did helped some students get closer to this thinking, especially when the final strips were added at the end. Oskar, for example, seemed unsure about the pattern he was seeing but was excited that he was able to predict what the next strips would say. He even said, "If we were dividing by 5, then the highest R we would get would be 4" (or something to that effect).

One idea Mr. Lerner had is that he could have introduced the strips by first revisiting the apple context more concretely. The class could have looked at a few strips in terms of groups of apples first together. This may have helped them recontextualize the 4 on their strips as a carton of apples and the quotient as how many cartons could be made. As the lesson stood, it wasn't until Josh cut the reflection short and asked students to look back at the apple cartons that it seemed to click for many that the reason the R never reached 4 was because a new carton of apples could be made. We are glad that he went back to the apples to close the lesson, but we think that this context could have been threaded throughout the lesson more coherently.

Participation: We were really proud of how kids participated in this lesson. We are always looking for the gender and racial distribution of the contributions to the discussion and we were happy to see that there were many girls who were driving the participation and sharing of ideas. Participation was also quite equitable across levels of math success (students who often struggle were able to come to the board or share their ideas quite frequently). Overall, kids seemed eager to talk and were confiding using the pointer to share their ideas as well as reflect on others' contributions.